Lesson 16. Assessing Multiple Linear Regression Models – Part 1

Note. In Part 2 of this lesson, you can run the R code that generates the outputs here in Part 1.

1 Overview

- Recall the multiple linear regression model

$$Y = \beta_0 + \beta_1 X_1 + \cdots \beta_k X_k + \varepsilon \quad \text{where} \quad \varepsilon \sim \text{iid } N(0, \sigma_\varepsilon^2)$$

- This is a population-level model
- We want to **infer** something about the population based on our sample
- Many of these upcoming inference topics will be familiar
  - We have seen them before in the context of simple linear regression

2 t-tests for coefficients

- Question: **Is an individual explanatory variable X_i helpful to include in the model, if the other explanatory variables are still there?**
- In other words: after we account for the effects of all the other predictors, does the predictor of interest X_i have a significant association with Y?
- Formal steps:
  1. State the hypotheses:

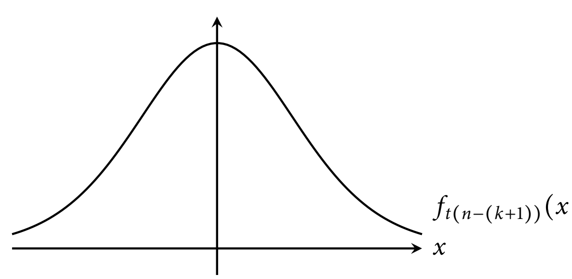
$$H_0 : \beta_i = 0$$
$$H_A : \beta_i \neq 0$$

  2. Calculate the test statistic:

$$t = \frac{\hat{\beta}_i}{SE_{\hat{\beta}_i}}$$

  3. Calculate the p-value:

     - If the conditions for multiple linear regression hold, then the the test statistic t follows $t(n-(k+1))$



4. State your conclusion, based on the given significance level α:

If we reject H_0 (p-value $\leq \alpha$):

We reject H_0 because the p-value is less than the significance level α. We see evidence that, after accounting for the other explanatory variables, X_i is significantly associated with Y.

If we fail to reject H_0 (p-value $> \alpha$):

We fail to reject H_0 because the p-value is greater than the significance level α. We do not see evidence that X_i is significantly associated with Y after accounting for the other explanatory variables.

The underlined parts above should be rephrased to correspond to the context of the problem

Example 1. After accounting for the size of a house, is its price related to its proximity to bike trails?

Use the `RailsTrails` data in the `Stat2Data` package to fit a multiple linear regression model predicting *Price2014* (price in thousands of dollars) from *SquareFeet* (size of house, in thousands of ft^2) and *Distance* (miles to nearest bike trail). Assume that the regression conditions are met.

We run the following R code:

```
fit <- lm(Price2014 ~ SquareFeet + Distance, data = RailsTrails)
summary(fit)
```

We obtain the following output:

```
Call:
lm(formula = Price2014 ~ SquareFeet + Distance, data = RailsTrails)

Residuals:
    Min      1Q  Median      3Q     Max
-152.15  -30.27   -4.14   25.75  337.93

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   78.985     25.607   3.085  0.00263 **
SquareFeet   147.920     12.765  11.588  < 2e-16 ***
Distance     -15.788      7.586  -2.081  0.03994 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 65.55 on 101 degrees of freedom
Multiple R-squared:  0.6574,  Adjusted R-squared:  0.6506
F-statistic: 96.89 on 2 and 101 DF,  p-value: < 2.2e-16
```

a. State the population-level model.

b. State the fitted model.

c. What do we learn from the estimated coefficient of *Distance*?

d. Is the association between *Distance* and *Price2014* statistically significant, after accounting for house size? Use a significance level of 0.05 to test whether the coefficient of *Distance* is 0. (Report the relevant values from the summary output.)

3 Confidence intervals for coefficients

- Goal: **We want to provide a range of plausible values for β_i, instead of just a point estimate**
- If the conditions for multiple linear regression are met, then we can form a $100(1-\alpha)\%$ CI for β_i with the following formula

$$\hat{\beta}_i \pm t_{\alpha/2,n-(k+1)} SE_{\hat{\beta}_i}$$

- Interpretation:

  We are 95% confident that the true coefficient of X_i is between lower endpoint of CI and upper endpoint of CI.

- Taking the interpretation even further:

  We are 95% confident that, holding the other explanatory variables constant, a one unit increase in X_i is associated with an average decrease/increase of between smaller magnitude of CI and larger magnitude of CI units in the response variable.

- The underlined parts above should be rephrased to correspond to the context of the problem

Example 2. Continuing with the `RailsTrails` example...

a. Based on the reported degrees of freedom for the residual standard error, what must n (the number of observations) be?

b. Use the R output to form a 95% confidence interval for the coefficient of *Distance*.
Note that $t_{0.05/2,101}$ = `qt(1 - 0.05/2, df = 101)` = 1.984.

c. Interpret your CI in the context of this problem.

We can also use the R function `confint()` to compute confidence intervals:

```
confint(fit, level = 0.95)
```

Here's the output:

A matrix: 3 × 2 of type dbl

	2.5 %	97.5 %
(Intercept)	28.18800	129.7824868
SquareFeet	122.59754	173.2421396
Distance	-30.83709	-0.7397968

4 ANOVA for multiple linear regression

- In addition to testing the individual explanatory variables one-by-one, we could also ask...
- Question: **Is the model as a whole effective?**
- In other words: is the model with all the explanatory variables better than a model with none of the explanatory variables?
- To answer this question, we return to the idea of partitioning variability:

$$SSTotal \quad = \quad SSModel \quad + \quad SSE$$

where

$$SSTotal = \sum_{i=1}^{n}(y_i - \bar{y})^2 \qquad SSModel = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 \qquad SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

5 The ANOVA table for multiple regression

Source	df	Sum of Squares	Mean Square	F-Statistic
Model				
Error				
Total				

Example 3. Continuing with the `RailsTrails` example...

Unfortunately, the R function `anova()` does not create the above ANOVA table. Instead, we can use the following R code to create the table manually:

```
y <- RailsTrails$Price2014
n <- 104
k <- 2

SSModel <- sum((predict(fit) - mean(y))^2)
SSE <- sum((y - predict(fit))^2)
SSTotal <- SSModel + SSE

MSModel <- SSModel / k
MSE <- SSE / (n - (k + 1))

F <- MSModel / MSE
```

Printing the values of `SSModel`, `SSE`, `SSTotal`, `MSModel`, `MSE`, and `F`, we see that their values are `832587.70`, `433959.48`, `1266547.18`, `416293.85`, `4296.63`, and `96.89`, respectively.

6 The ANOVA F-test for multiple linear regression

1. State the hypotheses:

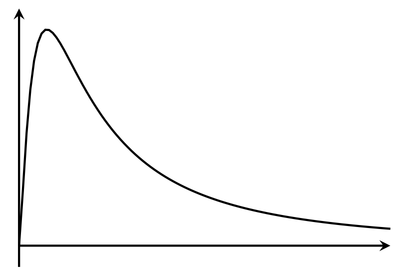
   Note that the alternative is not that every predictor has a non-zero coefficient

2. Calculate the test statistic:

$$F = \frac{MSModel}{MSE}$$

3. Calculate the p-value:

   - If the conditions for multiple linear regression hold, then the the test statistic F follows $F(k, n-(k+1))$



4. State your conclusion, based on the given significance level α:

   If we reject H_0 (p-value $\leq \alpha$):

   We see significant evidence that the model as a whole is effective.

   If we fail to reject H_0 (p-value $> \alpha$):

   We do not see sufficient evidence to conclude that the model is effective.

Example 4. Continuing with the `RailsTrails` example...

Perform an ANOVA F-test that determines whether the multiple linear regression model that uses *SquareFeet* and *Distance* to predict *Price2014* is effective as a whole.